$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2 \ldots\}$$

$\langle \mathbb{Z}, +, -, \cdot, / \rangle$; $\mathbb{Z}$ is closed with respect to $+, -, \cdot$ operations

$\mathbb{Z}$ – ring of integers

1. Closure $\quad +, -, \cdot$
2. Associativity $\quad \forall a, b, c \in \mathbb{Z} \rightarrow (a+b)+c = a+(b+c)$
   $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. „0" additively neutral element.
   $\forall a \in \mathbb{Z} : \quad a + 0 = 0 + a = a$
4. $\forall a \in \mathbb{Z} \rightarrow \exists! \, -a \in \mathbb{Z} : \quad a + (-a) = (-a) + a = 0$
   $-a$ is an additively inverse element.
5. „1" is a multiplicatively neutral element
   $\forall a \in \mathbb{Z} : \quad a \cdot 1 = 1 \cdot a = a$
6. ! Not all elements have multiplicatively inverse elem.
   such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$ except element 1.
7. Distribution property
   $\forall a, b, c \in \mathbb{Z} \rightarrow a \cdot (b + c) = a \cdot b + a \cdot c$

Algorithm in $\mathbb{Z}$:

1. Greatest Common Divider: >> gcd(a, n)

$\gcd(6, 15) = 3 \qquad \gcd(10, 15) = 5$

$\gcd(8, 15) = 1$

If $\gcd(a, n) = 1$, then a and n are relatively prime.

2. Extended Euklid Algorithm: >> eeuklid(a, n)

Operation modulo n: mod n.

Pvz. 1. $137 \bmod 11 = 5$

$137 = 12 \cdot 11 + 5$

$$\begin{array}{r|l} 137 & 11 \\ \underline{11} & 12 \\ 27 & \\ \underline{22} & \\ 5 & \end{array}$$

$137 = 12 \cdot 11 + 5$

$$\begin{array}{r} 11 \quad '12 \\ \hline 27 \\ 22 \\ \hline 5 \end{array}$$

Pvz. 2. $n=2$: $\forall a \in \mathbb{Z} \longrightarrow a \bmod 2 = \begin{cases} 0, & \text{if } a \text{ even} \quad (e) \\ 1, & \text{if } a \text{ odd} \quad (o) \end{cases}$

$a \bmod 2 \in \{0, 1\}$

$\mathbb{Z} \bmod 2 = \{0, 1\}$; $f_2 = \bmod 2 \longrightarrow f_2(\mathbb{Z}) = \{0, 1\} = \mathbb{Z}_2$

$f_2 : \mathbb{Z} \to \mathbb{Z}_2 = \{0, 1\}$

$\mathbb{Z}_2$ arithmetics : $< \mathbb{Z}_2, \underset{XOR}{\oplus}, \underset{AND}{\&} >$

+	e	o
e	e	o
o	o	e

$e \equiv 0$, $o \equiv 1$ $\longrightarrow$

$\oplus$	0	1
0	0	1
1	1	0

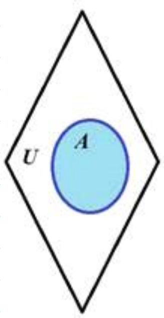
$\oplus$ XOR Exclusive OR

$\cdot$	e	o
e	e	e
o	e	o

$e \equiv 0$, $o \equiv 1$ $\longrightarrow$

&	0	1
0	0	0
1	0	1

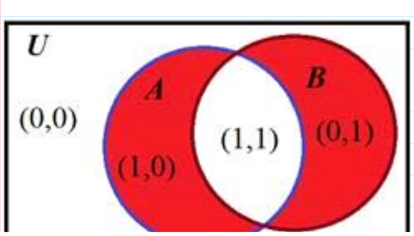
& AND Conjunction



XOR and AND logical operations in Boolean algebra can be illustrated by dartboard game.
Single Boolean variable can be represented by the set of 2 values {0,1} or {Yes,No} or {True,False}.
Let ***U*** is some universal set containing all other sets (we do not takke into account paradoxes related with ***U*** now).
Let ***A*** be a set in ***U***. Then with the set ***A*** in ***U*** can be associated a Boolean variable $\boldsymbol{b_A}$=1 if area ***A*** is hit by missile
$\boldsymbol{b_A}$=0 otherwise.

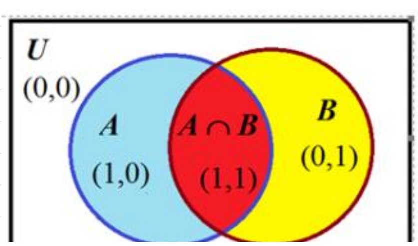
For this single variable b_A the negation (inverse) operation ` is defined:
$\boldsymbol{b}\neg_A$ =0 if $\boldsymbol{b_A}$=1,
$\boldsymbol{b}\neg_A$ =1 if $\boldsymbol{b_A}$=0.
Bollean operations are named also as Boolean functions.
Since negation operation/function is performed with the singe variable it is called a unary operation.

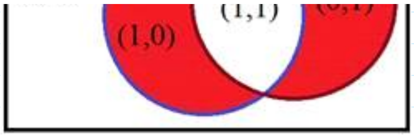
There are 16 Boolean functions defined for 2 variables and called binary functions.
Two of them XOR and AND are illustrated below.



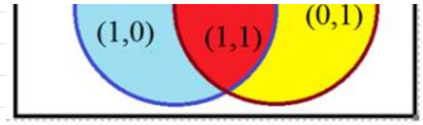
A	B	$A \oplus B$
0	0	0
1	0	1
0	1	1
1	1	0

A	B	A & B
0	0	0
1	0	0
0	1	0
1	1	1





1 1 | 0   1 1 | 1



Venn diagram of ***A*⊕*B*** operation.

Venn diagram of **A&B** operation.

$\langle \mathbb{Z}, +, -, * \rangle$; $\langle \mathbb{Z}_2, \oplus, \& \rangle$

$a \in \mathbb{Z}: a + 0 = a$; $a \in \mathbb{Z}_2: a \oplus 0 = a$; ? $a - a = 0$.

$a - a = a \oplus a = 0$; $a \oplus b \oplus a = b \oplus 0 = b$.

$\mathbb{Z}_3$ arithmetics : $\mathbb{Z} \bmod 3 = \mathbb{Z}_3 = \{0, 1, 2\}$

$(\mathbb{Z}_{30} = \{0, 3, 6, 9, \ldots\}) \bmod 3 = 0$

$(\mathbb{Z}_{31} = \{1, 4, 7, 10, \ldots\}) \bmod 3 = 1$

$(\mathbb{Z}_{32} = \{2, 5, 8, 11, \ldots\}) \bmod 3 = 2$

$\begin{array}{r|l} 9 & 3 \\ -9 & 3 \\ \hline 0 & \end{array}$ $\begin{array}{r|l} 7 & 3 \\ -6 & 2 \\ \hline 1 & \end{array}$ $\begin{array}{r|l} 11 & 3 \\ -9 & 3 \\ \hline 2 & \end{array}$

$\mathbb{Z} = \mathbb{Z}_{30} \cup \mathbb{Z}_{31} \cup \mathbb{Z}_{32}$; $\mathbb{Z}_{30}, \mathbb{Z}_{31}, \mathbb{Z}_{32}$ – are not intersecting

$\mathbb{Z}_n$ arithmetic $(n < \infty)$: $\mathbb{Z} \bmod n = \mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$, $0 \equiv n$

$\begin{array}{r|l} n & n \\ -n & 1 \\ \hline 0 & \end{array}$

$\mathbb{Z}_n$ is a ring with operations $+_{\bmod n}$ or $\cdot_{\bmod n}$

$\forall a, b \in \mathbb{Z}_n$: $a +_{\bmod n} b = c \in \mathbb{Z}_n$ Inverse operat.

$a \cdot_{\bmod n} b = d \in \mathbb{Z}_n$ $-_{\bmod n}$ $/ \bmod n$

$a + b = c \bmod n$

$a \cdot b = d \bmod n$

Operation properties:

$(a + b) \bmod n = (a \bmod n + b \bmod n) \bmod n$

$(a \cdot b) \bmod n = (a \bmod n \cdot b \bmod n) \bmod n$

$(a - b) \bmod n = \begin{cases} a - b, \text{ jei } a \geq b \\ a + n - b, \text{ jei } a < b \end{cases}$; $a, b < n$

$\begin{array}{r|l} n & n \\ -n & 1 \\ \hline 0 & \end{array}$ $0 \equiv n \bmod n$

For given $b \in \mathbb{Z}_n$. Find : $-b \in \mathbb{Z}_n$: $b + (-b) = 0 \in \mathbb{Z}_n$

$-b \bmod n = (0 - b) \bmod n = (n - b) \bmod n = n - b$

$(b + (-b)) \bmod n = (\cancel{b} + n \cancel{-b}) \bmod n = (0 + n) \bmod n = n \bmod n = 0$.

>> mb = mod(-b, n)

>> mod(b + mb, n) = 0

$>> mb = mod(-b, n)$

$>> mod(b + mb, n) = 0$

Let $n = p = 11$: $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$

Then $\mathbb{Z}_{11} = \{0, 1, 2, 3, \ldots, 10\}$; $+_{mod 11}$; $-_{mod 11}$; $*_{mod 11}$; $/_{mod 11}$

$\mathbb{Z}_p^* = \{1, 2, 3, \ldots, p-1\}$

Let we have any set ***G*** consisting of the elements of any nature, i.e. ***G***={***a***, ***b***, ***c***, …, ***z***, …}.

*1. **Definition**. A set **G** is an algebraic group if it is equipped with a binary operation • that satisfies four axioms:*

1. *Operation • is closed in the set;* for all ***a***, ***b***, there exists unique ***c*** in **G** *such that* ***a•b***=***c***.
2. *Operation • is associative; for all **a**, **b**, **c** in **G**:* (***a•b***)•***c***=***a•***(***b•c***).
3. *Group **G** has an neutral element abstractly we denote by **e** such that **a•e**=**e•a**.*
4. *Any element **a** in **G** has its inverse a^{-1} with respect to • operation such that* $a•a^{-1}= a^{-1}•a=e$ when ***e*** is neutral el.

For curiosity, can be said that group axioms seems very simple but groups and their mappings describes a very deep and fundamental phenomena in physics and other sciences. Among these mappings a special importance have mappings preserving operations from one group to another called isomorphisms, or homomorphisms and morphisms in general. Isomorphisms have a great importance in cryptography to realize a secure confidential ***cloud computing***. It is named as ***computation with encrypted data***. The systems having a homomorphic property are named as ***homomorphic cryptographic systems***. They are under the development and are very useful in creation of secure e-voting systems, confidential transactions in blockchain and etc. We do not present there the construction of these systems and postpone it to the further issues of BOCTII, say in BOCTII.2. There we present one very important isomorphism example later when consider so called discrete exponent function (DEF).

T1. Theorem. If p is prime, then $\mathbb{Z}_p^* = \{1, 2, 3, \ldots, p-1\}$ where operation is multiplication mod p is a multiplicative group.

Example : $p = 11 \Rightarrow \mathbb{Z}_{11}^* = \{1, 2, 3, \ldots, 10\}$

Multiplication Tab. Z_{11}*										
*	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	1	3	5	7	9
3	3	6	9	1	4	7	10	2	5	8
4	4	8	1	5	9	2	6	10	3	7
5	5	10	4	9	3	8	2	7	1	6
6	6	1	7	2	8	3	9	4	10	5
7	7	3	10	6	2	9	5	1	8	4
8	8	5	2	10	7	4	1	9	6	3
9	9	7	5	3	1	10	8	6	4	2
10	10	9	8	7	6	5	4	3	2	1

$2 \circ 6 = 12 \bmod 11 = 1$

$12 : 11 = 1$, $12 - 11 = 1$

$4 \cdot 3 \bmod 11 = 12 \bmod 11 = 1$

$4 \cdot 4^{-1} \bmod 11 = (4/4) = 1$

$\Downarrow$

$4^{-1} = 3 \bmod 11$

$5 \cdot 9 = 45 \bmod 11 = 1$

$5^{-1} \bmod 11 = 9$ $\quad 45 \mid 11$

9	9	7	5	3	1	10	8	6	4	2
10	10	9	8	7	6	5	4	3	2	1

$5 \cdot 9 = 45 \bmod 11 = 1$

$5^{-1} \bmod 11 = 9$

$$\begin{array}{r|l} 45 & 11 \\ \cline{2-2} 44 & 4 \\ \hline 1 & \end{array}$$

Power Tab. Z_{11}*										
^	0	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	4	8	5	10	9	7	3	6	1
3	1	3	9	5	4	1	3	9	5	4	1
4	1	4	5	9	3	1	4	5	9	3	1
5	1	5	3	4	9	1	5	3	4	9	1
6	1	6	3	7	9	10	5	8	4	2	1
7	1	7	5	2	3	10	4	6	9	8	1
8	1	8	9	6	4	10	3	2	5	7	1
9	1	9	4	3	5	1	9	4	3	5	1
10	1	10	1	10	1	10	1	10	1	10	1

The set of numbers that are generating all the numbers in the set Z_{11}^* is named as a set of generator $\Gamma_{11} = \{2, 6, 7, 8\}$

$\sim 40\%$ of Z_p^*

Let G be a finite group with $\mathrm{Card}(G) = |G| = N$.

Def. 1. The element g is a generator if g^i, $i = 0, 1, 2, N-1$, generates all N elements of G.

Def. 2. The group G which can be generated by generator g is a cyclic group and is denoted by $\langle g \rangle = G$.

Cyclic Group: $\mathbf{Z}_p^* = \{1, 2, 3, \ldots, p-1\}$; $\bullet_{\mathbf{mod}\ p}$, $:_{\mathbf{mod}\ p}$.

If $p = 11$, then

$q = (11-1)/2 = 5$

p, q are primes

Let p is prime.
Then p is **strong prime** if $p=2q+1$ where $q = (p-1)/2$ is prime as well.
Then g in $\mathbf{Z}_p^*$ is a generator of $\mathbf{Z}_p^*$ if and only if
(**iff**) $g^2 \neq 1$ **mod** p **and** $g^q \neq 1$ **mod** p.

For example, let p is strong prime and $p=11$, then one of the generators is $g=2$.
Verification method: $g^2 \neq 1$ **mod** p **and** $g^q \neq 1$ **mod** p.
The main function used in cryptography is Discrete Exponent Function - DEF:
$\mathrm{DEF}_g(x) = g^x \bmod p = a$.

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1 DEF v-4.pptx

Discrete Exponent Function DEF:

$\mathrm{DEF}_{p,g}(x) = g^x \bmod p = a$.

Power

$Z^* = \{1, 2, 3, \ldots 10\}$

$\mathrm{DEF}_{p,g}(x) = g \bmod p = a.$

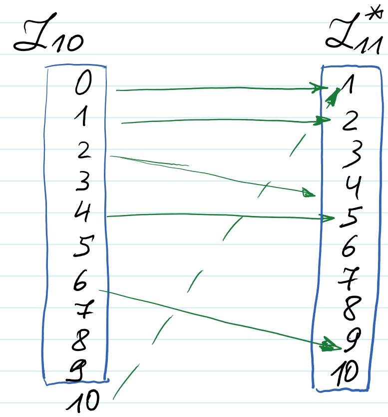
Power Tab. Z_{11}*						$x \in \mathbb{Z}_{10}$					
^	0	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	4	8	5	10	9	7	3	6	1
3	1	3	9	5	4	1	3	9	5	4	1
4	1	4	5	9	3	1	4	5	9	3	1
5	1	5	3	4	9	1	5	3	4	9	1
6	1	6	3	7	9	10	5	8	4	2	1
7	1	7	5	2	3	10	4	6	9	8	1
8	1	8	9	6	4	10	3	2	5	7	1
9	1	9	4	3	5	1	9	4	3	5	1
10	1	10	1	10	1	10	1	10	1	10	1

$\mathbb{Z}_{11}^* = \{1, 2, 3, \ldots, 10\}$

$\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\mathrm{DEF}: \mathbb{Z}_{10} \to \mathbb{Z}_{11}^*$

$\mathrm{DEF}_2(x) = 2^x \bmod 11 = a \in \mathbb{Z}_{11}^*$



T2. Fermat (little)Theorem. If ***p*** is prime, then [Sakalauskas, at al.]

$$z^{p-1} = 1 \bmod p$$

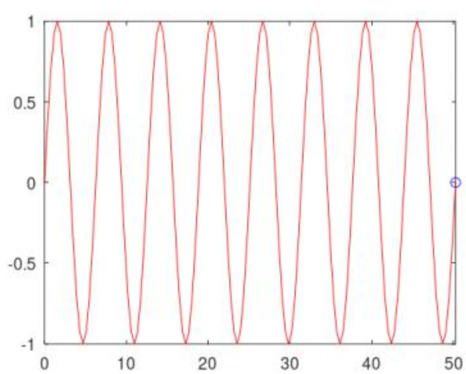
$z \in \mathbb{Z}_p^*$

$z^{p-1} = z^0 = 1 \bmod p$

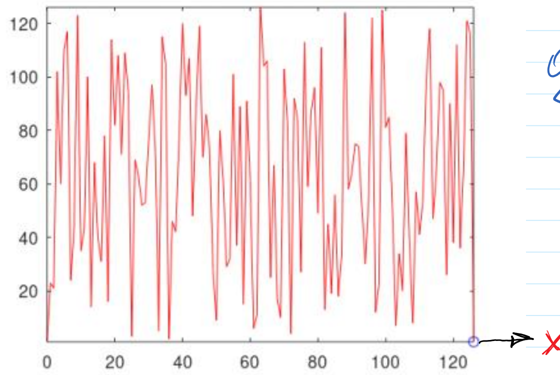
$z^k \bmod p = z^{k \bmod (p-1)} \bmod p$

$p-1 \equiv 0 \bmod (p-1)$

```
>> pi
ans = 3.1416
>> xrange=16*pi
xrange = 50.265
>> step=xrange/128
step = 0.3927
>> x=0:step:xrange;
>> y=sin(x);
>> comet(x,y)
```



```
>> p=127
p = 127
>> g = 23
g = 23
>> x=0:p-1;
>> a=mod_expv(g,x,p)
>> comet(x,a)
```



$g^x \bmod p = a$

$$\left.\begin{array}{l} \mathrm{card}(\mathcal{I}_{10}) = |\mathcal{I}_{10}| = 10 \\ \mathrm{Card}(\mathcal{I}_{11}^*) = |\mathcal{I}_{11}^*| = 10 \end{array}\right\} \Rightarrow \mathrm{card}(\mathcal{I}_{10}) = \mathrm{card}(\mathcal{I}_{11}^*)$$

$g^x \bmod p = a$

It is proved that:
if p is prime, then there exists such numbers g that $DEF_g(x)$ provides 1-to-1 or bijective mapping.

Security considerations: if someone can comput for example a secret param. x generated by A the he/she can compute secret param x if a, p and g are given.

Adv.: $g^x \bmod p = a$

If p is generated large enough, e.g. $p \sim 2^{2048} \approx 10^{670}$, $|p| = 2048$ bits the to find x when p, g and a are given is infeasible with classical computers.

It is ifeasible to compute x from the equation $g^x \bmod p = a$ by having p, g and a.

The probleem to find x when p, g and a are given is called a discret logarithm probleem — DLP

$$d\log_g (g)^x \bmod p = x \cdot d\log_g (g) \bmod p = x \cdot 1 \bmod p = x.$$

OWF
One-way-functions: Discrete Exponent Function (DEF) is a conjectured (OWF)

One - way - functions : Discrete Exponent Function (DEF)
is a conjectured (OWF)

1) It is easy to compute $a = g^x \bmod p$, when x, g, p are given.

2) It is infeasible to find x satisfying the condition $a = g^x \bmod p$ when a, g, p are given.

Yao theorem: if pseudo random numbers generators exist $\Leftrightarrow$ OWFs exist & vise versa!

How to find inverse element to z mod n?
>> mulinv(z,n)

Inverse elements in the Group of integers <$\mathbf{Z}_p^*$, • **mod** ***p***> can be found using either Extended Euclidean algorithm or Fermat theorem, or ...

Let we have ***z*** in $\boldsymbol{Z}_p^*$, then to find z^{-1} mod ***p*** it can be done by Octave:
>> z_m1=mulinv(z,p)

$z \in \mathbb{Z}_p^*$: to find z^{-1} such that $z \cdot z^{-1} = z^{-1} * z = 1 \bmod p$

$$z^{p-1} = 1 \bmod p \;/ \cdot z^{-1} \Rightarrow z^{p-1} \cdot z^{-1} = z^{-1} \bmod p \Rightarrow$$
$$\Rightarrow z^{-1} = z^{p-1} \cdot z^{-1} \bmod p \Rightarrow z^{-1} = z^{p-2} \bmod p$$

$$z^{-1} = z^{p-2} \bmod p$$

Operations in exponents.

$$\left.\begin{aligned} a^r \cdot a^s \bmod p &= a^{(r+s) \bmod (p-1)} \bmod p \\ (a^r)^s \bmod p &= a^{(r \cdot s) \bmod (p-1)} \bmod p \end{aligned}\right\}$$ According to Fermat th. we have:

$$\left.\begin{aligned} z^0 &= 1 \bmod p \\ z^{p-1} &= 1 \bmod p \end{aligned}\right\} \Rightarrow 0 \equiv p-1 \text{ in exponents } 0 \equiv p-1 \bmod (p-1)$$

Let we need to compute expression: $g^{s \bmod (p-1)} \bmod p$

where s is in exponent of the generator g,
when $s=(i+x\cdot h) \bmod (p-1)$; $r=g^i \bmod p$.
$\sigma=(r,s)$

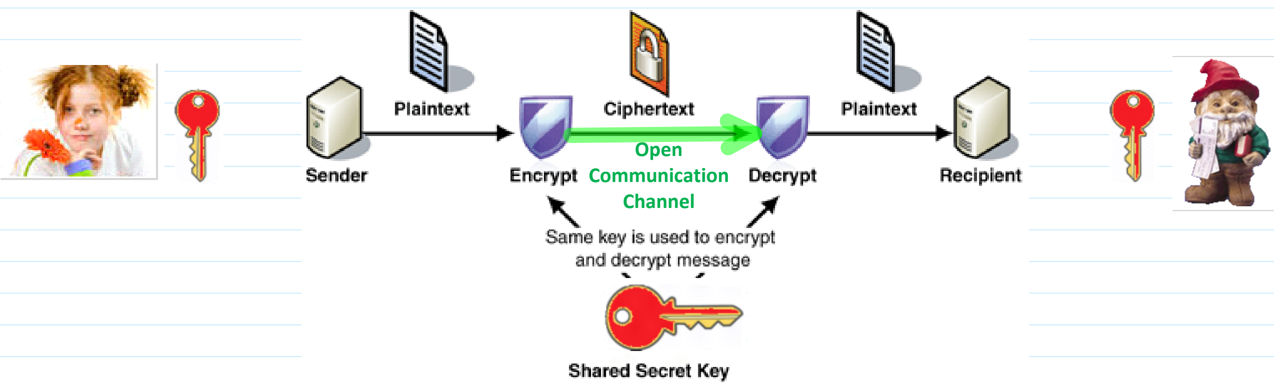
$$g^{s \bmod (p-1)} \bmod p = g^{(i+x\cdot h) \bmod (p-1)} \bmod p = g^i \cdot (g^x)^h = r\cdot a^h \bmod p.$$

Discrete exponent function :
$a = g^{x} \bmod p$; $p \sim 2^{\underline{2048}} \approx 10^{700}$

$>> a = mod\_exp(g, x, p)$

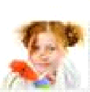
```
>> mod_exp(2,3,7)
ans = 1
```

We will deal with integers of 28 bits
$p \sim 2^{28}-1$



Diffie-Hellman Key Agreement Protocol (DH KAP)

Public Parameters PP=(p,g)



$u \leftarrow rand(Z_p^*)$
$g^u \bmod p = t_A$ $\longrightarrow$ t_A
t_B $\longleftarrow$

$v \leftarrow rand(Z_p^*)$
$t_B = g^v \bmod p$

$k_{AB} = (t_B)^u \bmod p =$
$=(g^v)^u \bmod p = g^{vu} \bmod p$

$k_{BA} = (t_A)^v \bmod p =$
$=(g^u)^v \bmod \; = g^{uv} \bmod p$

$k_{AB} = k = k_{BA}$

Security considerations : if someone can compute for example a secret param. u generated by A the he/she can compute secret key k by intercepting t_B

Adv.: $(t_B)^u \bmod p = k$.

If p is generated large enough, e.g. $p \sim 2^{2048} \approx 10^{700}$, $|p| = 2048$ bits the to find u when p, g and t_A are given is infeasible with classical computers.

It is ifeasible to compute u from the equation $g^u \bmod p = t_A$ by having p, g and t_A.

The problem to find u when p, g and t_A are given is called a discret logarithm problem - DLP

$$\mathrm{dlog}_g (g)^u \bmod p = u \cdot \mathrm{dlog}_g (g) \bmod p = u \cdot 1 \bmod p = u.$$